

用四阶 Runge-Kutta 法和 TVD 格式 求解二维跨超音流场

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【摘要】 利用坐标转换给出二维任意曲线坐标系下雅可比矩阵特征值、特征向量及 Harten 型 TVD 格式的详细表达式,并结合四阶 Runge-Kutta 法发展了一种新的、有效的求解超音、跨音流的算法。数值计算表明,本文的方法可有效地求解跨音流场。

主题词: TVD 格式 四阶 Runge-Kutta 法 Euler 方程组 超跨音流场
分类号: O354

1 二维任意曲线坐标系下 TVD 格式

在已公开发表的文献中,大都只给出直角坐标系下 TVD 格式的详细表达式^[1],不便于直接用于复杂区域问题的求解。在时间相关法解流场的数值计算中,隐式格式的 CFL 数虽可取大,但对方程组求解时,会出现块三角矩阵求解问题,处理比较麻烦,而利用通常的显式格式求解,虽可直接对方程组求解,但计算时间较长。本文尝试将四阶 Runge-kutta 法和 TVD 格式结合起来,求解二维超、跨音流场。

二维任意曲线坐标系下 Euler 方程组的矩阵形式为:

$$\frac{\partial Q}{\partial t} + \frac{\partial E}{\partial \xi} + \frac{\partial F}{\partial \eta} = 0 \quad (1)$$

$$Q = \frac{1}{J} \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ E_t \end{pmatrix} \quad E = \frac{1}{J} \begin{pmatrix} \rho G_1 \\ \rho u G_1 + \zeta_p \\ \rho v G_1 + \zeta_p \\ (E_t + p) G_1 \end{pmatrix} \quad F = \frac{1}{J} \begin{pmatrix} \rho G_2 \\ \rho u G_2 + \eta_p \\ \rho v G_2 + \eta_p \\ (E_t + p) G_2 \end{pmatrix}$$

$$J = \zeta \eta_\eta - \eta \zeta_\zeta \quad G_1 = \zeta u + \zeta v \quad G_2 = \eta u + \eta v$$

$E_t = \rho [e + 1/2(u^2 + v^2)]$, $p = \rho e(\gamma - 1)$, γ 为气体绝热指数。

令 $A = \partial E / \partial Q = R \Lambda R^{-1}$; $\Lambda_i = \lambda^i \delta_i$, (i 不求和); A 为雅可比矩阵, R 和 R^{-1} 为左、右特征向量, λ^i 为特征值。变化(1)式有: $\partial Q / \partial t + R \Lambda R^{-1} \partial Q / \partial \xi + \partial F / \partial \eta = 0$ 。令 $R^{-1} Q = W = (w^1, w^2, w^3, w^4)^T$, 有: $\partial W / \partial t + \Lambda \partial W / \partial \xi + R^{-1} \partial F / \partial \eta = 0$ 。令 $\Delta_{j+1/2} Q = Q_{j+1,k} - Q_{j,k}$, $\Delta_{j+1/2} Q = \sum_{i=1}^m \alpha_{i+1/2}^j R_{j+1/2}^i$ 。对(1)式作用系数矩阵:

$$Te = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

得:

$$\partial Q / \partial t + \partial E / \partial \xi + \partial F / \partial \eta = 0 \quad (2)$$

$$\bar{Q} = TeQ = \frac{1}{J} \begin{pmatrix} \rho \\ \rho \bar{G}_1 \\ \rho \bar{G}_2 \\ E_t \end{pmatrix} \quad \bar{E} = TeE = \frac{1}{J} \begin{pmatrix} \rho \bar{G}_1 \\ \rho \bar{G}_1 \bar{G}_1 + p \\ \rho \bar{G}_1 \bar{G}_2' \\ (E_t + p) \bar{G}_1 \end{pmatrix}$$

$\bar{G}_1 = (u\zeta + v\bar{\zeta}) / |\nabla \bar{\zeta}|$, $\bar{G}_2' = (-u\zeta + v\bar{\zeta}) / |\nabla \bar{\zeta}|$, $\cos\theta = \zeta / |\nabla \bar{\zeta}|$, $\sin\theta = \bar{\zeta} / |\nabla \bar{\zeta}|$
 相比于笛卡尔坐标系下欧拉方程组, $\bar{Q}, \bar{E}$ 有相同的形式, 类似于文献[2], 可以得到特征向量:

$$R = \begin{bmatrix} 1 & 1 & 1 & 0 \\ u - c\cos\theta & u & u + c\cos\theta & -\sin\theta \\ v - c\sin\theta & v & v + c\sin\theta & \cos\theta \\ H - \bar{G}_1 c & (u^2 + v^2)/2 & H + \bar{G}_1 c & \bar{G}_2' \end{bmatrix} \quad (3)$$

$$R^{-1} = \begin{bmatrix} \frac{1}{2} \left(b_1 + \frac{\bar{G}_1}{c} \right) & \frac{1}{2} \left(-b_2 u - \frac{1}{c} \cos\theta \right) & \frac{1}{2} \left(b_2 v - \frac{1}{c} \sin\theta \right) & \frac{1}{2} b_2 \\ \frac{1}{2} \left(b_1 - \frac{\bar{G}_1}{c} \right) & \frac{1}{2} \left(-b_2 u + \frac{1}{c} \cos\theta \right) & \frac{1}{2} \left(b_2 v + \frac{1}{c} \sin\theta \right) & \frac{1}{2} b_2 \\ -\bar{G}_2' & -\sin\theta & \cos\theta & 0 \end{bmatrix} \quad (4)$$

特征值 $\{\lambda\} = (\bar{G}_1 - c, \bar{G}_1, \bar{G}_1 + c, \bar{G}_1) / |\nabla \bar{\zeta}|$, 于是有:

$$\bar{\alpha}_{j+1/2}^l = R_{j+1/2}^{-1} \Delta_{j+1/2} Q \quad (5)$$

相应的二阶精度 TVD 格式为:

$$E_{j+1/2, k} = \frac{1}{2} (E_{j, k} + E_{j+1, k}) + \frac{1}{2\lambda} \sum_{l=1}^m [g_j^l + g_{j+1}^l - Q(\bar{U}_{j+1/2} + Y_{j+1/2}^l) \bar{\alpha}_{j+1/2}^l] R_{j+1/2}^l \quad (6)$$

$$Y_{j+1/2}^l = \begin{cases} (g_{j+1}^l - g_j^l) / \bar{\alpha}_{j+1/2}^l & \bar{\alpha}_{j+1/2}^l \neq 0 \\ 0 & \bar{\alpha}_{j+1/2}^l = 0 \end{cases}$$

$$g_j^l = \min \text{mod}(\tilde{g}_{j-1/2}^l, \tilde{g}_{j+1/2}^l)$$

$$\tilde{g}_{j-1/2}^l = \frac{1}{2} [Q(\bar{U}_{j+1/2}) - \bar{U}_{j+1/2}^2] \bar{\alpha}_{j+1/2}^l$$

$$\min \text{mod}(x, y) = \max[0, \text{sgn}(x) \cdot \min(|x|, |y|)]$$

$$\bar{U}_{j+1/2}^l = \frac{\Delta t}{\Delta x} \lambda_{j+1/2}^l$$

$$Q(x) = \begin{cases} x & |x| \geq \epsilon \\ (x^2 + \epsilon^2) / 2\epsilon & |x| < \epsilon \end{cases}$$

ϵ 为一给定的小正数。min mod 函数的引入保证了格式具有 TVD 性质, 从而可有效地抑制数值振荡, $Q(x)$ 可视为数值粘性系数, 可在 $\bar{U} = 0$ 时增加数值耗散, 以避免非物理解的出现。

上述式中, c 是局部音速, H 是总焓, $b_1 = b_2(u^2 + \bar{v}^2)/2$, $b_2 = (Y - 1)/c^2$ 。类似地有 $F_{j, k+1/2}$ 的 TVD 和表达式。

2 四阶 Runge-Kutta 法

对(1)式取守恒型差分格式有:

$$\begin{aligned} & \Delta Q_{j,k}^n + \frac{\lambda \zeta \theta}{1 + \omega} (E_{j+1/2,k}^{n+1/2} - E_{j-1/2,k}^{n+1/2}) + \frac{\lambda \eta \theta}{1 + \omega} (F_{j,k+1/2}^{n+1/2} - F_{j,k-1/2}^{n+1/2}) \\ & = - \frac{\lambda \zeta (1 - \theta)}{1 + \omega} (E_{j+1/2,k}^n - E_{j-1/2,k}^n) - \frac{\lambda \eta (1 - \theta)}{1 + \omega} (F_{j,k+1/2}^n - F_{j,k-1/2}^n) \end{aligned} \quad (7)$$

上式 n 表示叠代层次, $\Delta Q_{j,k}^n = Q_{j,k}^{n+1} - Q_{j,k}^n$, 当 $\theta = \omega + 1/2$ 时, (7) 式时间上是二阶精度, 空间精度由 $E_{j+1/2,k}$ 和 $F_{j,k+1/2}$ 的表达式确定。 $\theta = 1$ 为全隐格式, $\theta = 0$ 为全显格式。

自 Jameson & Schmidt^[4] 以来, 不少文献^[5]用四阶 Runge-Kutta 法求解流场问题, 本文对 $E_{j+1/2,k}$ 取(6)式, 即结合 TVD 格式求解流场。具体实施过程:

$$\text{取} \quad R = - \left[\frac{\partial E}{\partial \zeta} + \frac{\partial F}{\partial \eta} \right] = - \left[\frac{E_{j+1/2,k} - E_{j-1/2,k}}{\Delta \zeta} + \frac{F_{j,k+1/2} - F_{j,k-1/2}}{\Delta \eta} \right]$$

$$\begin{aligned} \text{有:} \quad Q^1 &= Q^n + (1/4) \Delta t R(Q^n), \quad Q^2 = Q^n + 1/3 \Delta t R(Q^1) \\ Q^3 &= Q^n + (1/2) \Delta t R(Q^2), \quad Q^{n+1} = Q^n + \Delta t R(Q^3) \end{aligned}$$

3 算例分析:

算例 1 矩形通道的斜激波反射。该算例来源于 H. C. Yee^[2], 激波具有激波角 29° ; 进口马赫数 $Ma_{in} = 2.9$, 截面为 $4.1 \text{ m} \times 1.0 \text{ m}$ 的矩形, 计算网格由 Thompson 法生成, 固壁处为滑移边界条件, 出口处外推边界条件, 得出图 1 和图 2 的等压力图和中心高度压力系数线。同文献 [2] 相比较, 可知本文用坐标转换得出的任意曲线坐标系下雅可比矩阵的特征值、特征向量及相应的 TVD 格式详细表达式是正确的。在本算例中, 取 $CFL = 0.8$, 迭代 300 步收敛。因而用四阶 Runge-Kutta 法结合 TVD 格式计算是可行的。

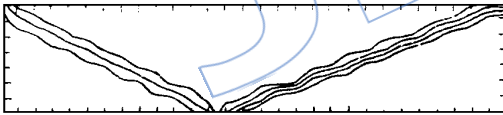


图 1 斜激波反射等压力图

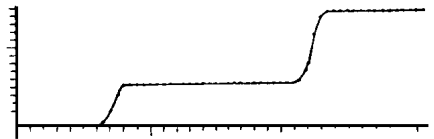


图 2 斜激波反射中间截面压力系数

算例 2 楔体激波生成及反射。算例取自 E. Venkatapathy et al.^[6]。得出图 3 和图 4 的等压力图和中心面压力系数线。从图 1 到图 4 可以看出, TVD 格式用于求解激波问题, 具有分辨率高(激波厚度在 3 个网格点内), 波动很小等特点。尤其在激波反射区, 其 CFL 数仍可取接近 1 的值, 这一点是同一精度矢量通量分裂法所不具备的^[9]。

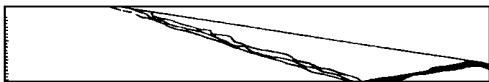


图 3 楔体流场等压图

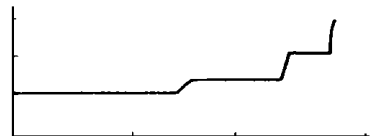


图 4 中间截面压力系数

算例 3 跨音喷管的计算。算例来源 Mason et al.^[7], 进口边界条件 $Ma_{in} = 0.232$, 背压与进

口滞止压力之比为 0.1135, 固壁取滑移边界条件。从图 5- 图 7 中可以看出, TVD 格式相比于

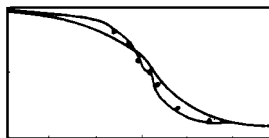


图 5 喷管中间截面压力

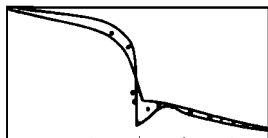


图 6 喷管壁面压力

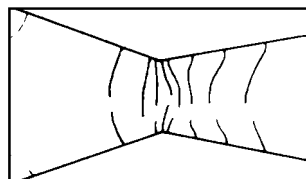


图 7 马赫数(TVD 格式)

SIMPLE^[8] 法求解跨音流场更接近实验值, 这可能是文献[8]采用混合格式精度不足二阶造成的。因而用 TVD 格式求解跨音流场也是可行的。

4 结 论

(1) 本文给出的任意曲线坐标系下的 TVD 格式详细表达式是正确的, 所得结果是可靠的。(2) 将 TVD 格式和四阶 Runge-Kutta 法结合使用, 具有求解方便, 耗时不多等优点, 是一种值得推广的算法。(3) Harten 型 TVD 格式是一种耗散型差分格式, 也可用于跨音流场计算。

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flow on the stall flutter has been investigated experimentally. The results of experiments show that the small radial distortion of the inlet flow at blade tip can eliminate the stall flutter. This phenomenon is verified by computation with three-dimensional actuator disk method. The computed results conform with the experimental data.

Key words Axial-flow compressors Radial distortion Flutter

NUMERICAL SIMULATION OF INTERNAL FLOW IN 2- D NOZZLES WITH THRUST VECTORING

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ABSTRACT Numerical simulation of internal flow with thrust vectoring in two-dimensional convergent nozzles is described, which utilizes multigrid technique in conjunction with MacCormack's explicit scheme. In computations, the Reynolds averaged Navier-Stokes equations are adopted for the five grids with $K-\epsilon$ turbulence model; and on the other hand, Euler equations are solved on the coarse grids. Numerical results provide good resolution to the flow fields, and the computed pressure distributions are found in good agreement with the experimental data. The influences of geometric parameters upon the flow quality and thrust coefficient are also analyzed. It is demonstrated that the multigrid technique adopted here is reliable and suitable for parametric study.

Key words Jet nozzles Thrust vector control Numerical simulation

COMPUTATION OF 2- D TRANSONIC AND SUPERSONIC FLOW FIELDS USING TVD SCHEME AND FOUR- STAGE RUNGE- KUTTA TECHNIQUE

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ABSTRACT Harten's TVD scheme is extended to 2- D general curvilinear coordinates by local rotation matrix translation. A new fully explicit solving method based on Four- Stage Runge- Kutta technique is developed to predict 2- D supersonic and transonic flow field. Comparison between computation and experiment is presented for both supersonic and transonic flow. The numerical results show that this scheme is not only convenient to solve the supersonic flow, but also capable of predicting the transonic flow.

Key words Euler motion equation Supersonic flow Transonic flow

INVESTIGATION ON FLOW RESISTANCE AND HEAT TRANSFER IN COOLING PASSAGES OF GAS TURBINE BLADE

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ABSTRACT An enlarged test blade of gas turbine has been investigated experimentally for flow resistance and heat transfer in its cooling passages. Air entered the radial cooling channel inside the leading part of the blade and contrary flowed downward into the passage between two plates while the air flow reached to the top. The bulk of the air stream